APPLICATION OF THE PARTITION OF UNITY FINITE ELEMENT METHOD TO EXTERIOR ACOUSTICS PROBLEMS

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ABSTRACT

The Partition of Unity Finite Element Method (PUFEM) is dedicated in standard acoustics to high frequency problems or large dimensions problems. Its main feature is indeed to capture several wavelengths per element with a very high convergence rate. The modeling of acoustics waves in exterior unbounded domains seems therefore adapted for the PUFEM. However non reflecting boundary conditions are necessary to handle this task. Some analytical boundary conditions have already been tested with the PUFEM. Here we propose to extend the choice of possible non reflecting boundary conditions in the PUFEM with the Perfectly Matched Layers (PML).

1 FORMULATION

1.1 Non Reflecting Boundary Conditions (NRBC)

The following weak form of the Helmholtz equation is used in the finite element method applied to acoustics :

$$\int_\Omega (\nabla p \cdot \nabla \delta p - k^2 p\, \delta p)\, \mathrm{d}\Omega - \int_\Gamma \frac{\partial p}{\partial n}\, \delta p\, \mathrm{d}\Gamma = 0\ . \tag{1}$$

It describes the behavior of the acoustic pressure p in a fluid domain Ω at an angular frequency ω with a wavenumber k. Here the exterior domain Ω is given with rigid boundaries Γ_r and non reflecting boundaries Γ_∞. To handle these last boundaries Laghrouche et al. [1] tested and compared some Non Reflecting Boundary Conditions (NRBC) such as Robin type boundary conditions, exact boundary conditions (DtN) and approximate NRBCs (Bayliss, Gunzburger and Turkel - Engquist and Majda - Feng). In the following we propose to extend the choice of possible NRBCs in the PUFEM with the Perfectly Matched Layers (PML). The idea behind the PML is to stretch the coordinates in the complex domain to get an absorbing domain. Along the x-axis for example, a plane wave $\exp(\mathrm{i}(kx - \omega t))$ becomes $\exp(\mathrm{i}(k\tilde{x} - \omega t))$ with $\tilde{x} = x + \mathrm{i}f(x)$. In the absorbing region of this PML the wave decays exponentially. By choosing also $\mathrm{d}f/\mathrm{d}x = \sigma_x(x)/\omega$, the attenuation rate becomes frequency independent. Note that the function $\sigma_x(x)$ cannot be a simple large constant since it would lead to numerical reflections at the end of the PML domain. However the larger the value of the integral $\int_{PML} \sigma_x(x)\mathrm{d}x$, the best. In order to achieve that goal, Bermúdez[2] tried unbounded functions such as $\int_{PML} \sigma_x(x)\mathrm{d}x = +\infty$ and concluded that the optimal absorbing function was $\sigma_x(x) = c\,(L_x - x)^{-1}$. In the following we keep this same function in both directions x and y.

In practice, the complex stretching of our original differential Equation (1) writes :

$$\int_\Omega (\frac{\gamma_y}{\gamma_x}\frac{\partial p}{\partial x}\frac{\partial \delta p}{\partial x})\, \mathrm{d}x\mathrm{d}y + \int_\Omega (\frac{\gamma_x}{\gamma_y}\frac{\partial p}{\partial y}\frac{\partial \delta p}{\partial y})\, \mathrm{d}x\mathrm{d}y - \int_\Omega (k^2\gamma_x\gamma_y p\delta p)\, \mathrm{d}x\mathrm{d}y - \int_\Gamma \frac{\partial p}{\partial n}\, \delta p\, \mathrm{d}\Gamma = 0\ , \tag{2}$$

with $\gamma_x(x) = 1 + \mathrm{i}\frac{\sigma_x(x)}{\omega}$ and $\gamma_y(y) = 1 + \mathrm{i}\frac{\sigma_y(y)}{\omega}$.

1.2 Partition of Unity Finite Element Method (PUFEM) in 2D

The key ingredient of the PUFEM relies on the enrichment of the conventional finite element approximation by including solutions of the homogeneous partial differential equation ([3, 4]). In this work, plane waves are chosen for the enrichment. In each sub-domain, the acoustic pressure is hence expanded as

$$p(\boldsymbol{r}) = \sum_{j=1}^{3}\sum_{q=1}^{Q_j^{(k)}} N_j^3(\xi,\eta)\exp\left(\mathrm{i}k\boldsymbol{d}^{(k)}\cdot(\boldsymbol{r}-\boldsymbol{r}_j^{(k)})\right)A_{jq}^{(k)}, \tag{3}$$

where the plane waves amplitudes $A_{jq}^{(k)}$ are unknown coefficients and functions N_j^3 are the classical linear shape functions on triangular elements. Points $\boldsymbol{r}_j^{(k)}$ are the nodes associated with element $V^{(k)}$. The directions are chosen to be evenly distributed over the unit circle, that is

$$\boldsymbol{d}^{(k)} = (\cos(\theta_q), \sin(\theta_q)) \quad \text{where} \quad \theta_q = \frac{2\pi q}{Q_j^{(k)}}, \quad q = 1, \ldots, Q_j^{(k)}. \tag{4}$$

The number of plane waves attached to each node $j = 1, 2, 3$ depends on the frequency and the element size according to the following criteria [5] :

$$Q_j^{(k)} = \mathrm{round}[kh + C(kh)^{1/3}]. \tag{5}$$

Here, h is taken as largest element edge length connected to node j within the acoustic domain and the constant C is usually chosen to lie in the interval $C \in [2, 20]$. This coefficient can be adjusted depending on the configuration and the expected accuracy.

In the present work, the finite element geometries are defined using standard quadratic shape functions on triangular elements :

$$\boldsymbol{r}^{(k)}(\xi, \eta) = \sum_{j=1}^{6} N_j^6(\xi, \eta)\boldsymbol{r}_j^{(k)}, \tag{6}$$

as this description is integrated in most softwares (here the finite element mesh generator `Gmsh` is used). In Eq. (6) extra nodes $\boldsymbol{r}_j^{(k)}$ for $j = 4, 5, 6$ correspond to the mid-node of the edges.

2 RESULTS

Figure (1) presents the real part of the pressure radiated at 5000 Hz in a semi-infinite domain by a point source located at 5cm above an infinite plane. The acoustic domain spans over a 2 meters square while the added PML has a thickness of 0.5m. The result, obtained with an error $\mathcal{L}_2$ lower than 5% compared to analytical results, shows the good efficiency of the PML coupled to the PUFEM.

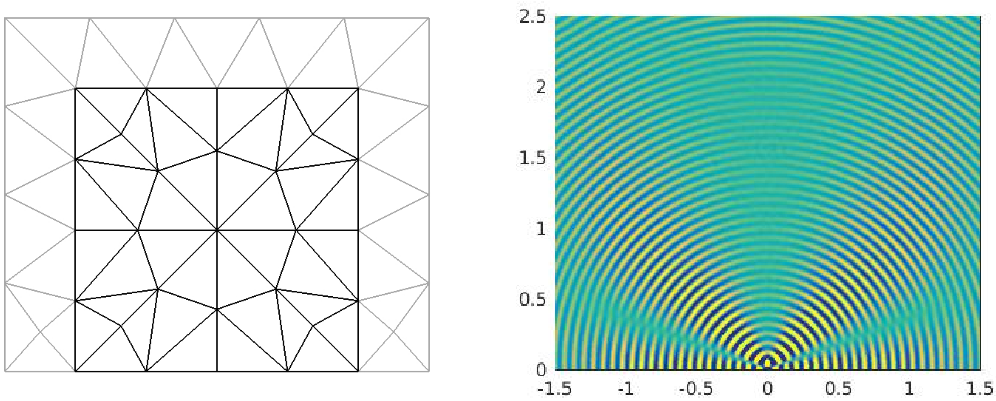

Figure 1: Point source radiating in a semi-infinite domain. From left to right : PUFEM mesh, real part of the pressure.

Figure (2) presents the real part of the pressure radiated at 1000 Hz in a semi-infinite domain by an imposed harmonic displacement over a circle. The acoustic domain spans over a disk of radius 5 with an included 0.5m thick PML. This result is an other illustration of the good efficiency of the PML coupled to the PUFEM.

3 CONCLUSION

In conclusion the stretching in the complex domain of the acoustic problem in order to create an absorbing layer called a PML works well with the PUFEM. It offers an other way to model non reflecting boundary conditions in the PUFEM, easy to implement in any configuration. The

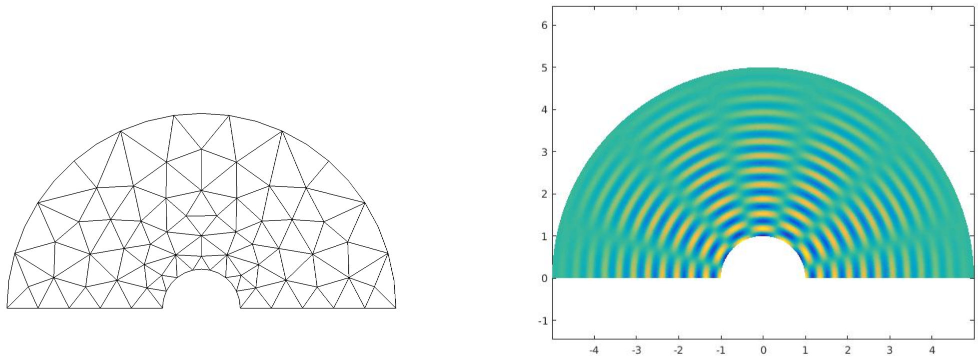

Figure 2. Imposed displacement - Radial PML for $R \in [2.5; 5]$.

PUFEM coupled with the PML offers hence an efficient tool to model the propagation and the scattering of acoustic waves at high frequency in exterior problems of large dimensions.

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