TIME DOMAIN FINITE ELEMENT ANALYSIS OF STRUCTURES WITH FRACTIONAL VISCOELASTIC DAMPING USING TIME-DIFFUSIVE SCHEME

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ABSTRACT

This work aims at simulating the time response of a structure damped by viscoelastic materials. The structure is discretised by finite elements and a 4-parameter fractional derivative model is used to describe the frequency-dependency of the mechanical properties of the viscoelastic material. The proposed approach combines a classical Newmark time-integration scheme to solve the semi-discretised equation of motion with a diffusive representation of fractional derivatives. This approach is applied to a finite element model, and validated on a single degree-of-freedom system for which an analytical solution can be derived.

1 INTRODUCTION

The importance of fractional calculus for modeling viscoelastic material behavior has been recognized by the mechanical scientific community since the pioneering work of Bagley and Torvik [1]. The merits of using fractional differential operator lie in the fact that few parameters are needed to accurately describe the constitutive law of damping materials and the resulting model can be easily fitted to experimental data over a broad range of frequencies. While the use of such models is quite straightforward in the frequency domain, some difficulties arise from their application in the time-domain, due to the presence of fractional derivatives.
The resolution methods are classically either based on time discretization of the fractional dynamics (see e.g. [2]), or on diffusive representations (cf. [3]). For large scale systems, the first method proves memory consuming because it is necessary to store the whole displacement history of the system due to the non-local character of the fractional derivatives. The second method, based on diffusive realizations of fractional derivatives, is numerically more efficient because it has no hereditary behavior, thus avoiding the storage of the solution from all past time steps. The diffusive representation, coupled with a Newmark integration scheme, has already been developed and validated for a fractionally damped single-of-freedom system [4]. In [5], an extension of this approach to viscoelastic structures using FE modeling and a fractional derivative model has been presented but not tested. The purpose of this work is to implement the method described in [5] and to apply it to a structure with viscoelastic damping. The approach is validated on a single degree-of-freedom system for which an analytical solution can be derived.

2 FINITE ELEMENT VISCOELASTIC PROBLEM

We consider a structure composed of elastic and viscoelastic materials. A fractional derivative model is identified to described the frequency-dependency of the complex shear and the bulk moduli (resp. $\hat{G}$ and $\hat{K}$) of the viscoelastic material:

$$\hat{G}(\omega) = G_0 + \frac{(G_\infty - G_0)(\mathrm{i}\omega\tau_G)^{\alpha_G}}{1 + (\mathrm{i}\omega\tau_G)^{\alpha_G}} \quad \text{and} \quad \hat{K}(\omega) = K_0 + \frac{(K_\infty - K_0)(\mathrm{i}\omega\tau_K)^{\alpha_K}}{1 + (\mathrm{i}\omega\tau_K)^{\alpha_K}} \tag{1}$$

where G_0 and K_0 are relaxed moduli, G_∞ and K_∞ are unrelaxed moduli satisfying $G_\infty > G_0$ and $K_\infty > K_0$, $\tau_G > 0$ and $\tau_K > 0$ are relaxation times and α_G and α_K are fractional coefficients comprised bewteen 0 and 1. Figure 1 shows that the fractional derivative model enables a good representation of the frequency-dependency both the shear and the bulk moduli over a wide frequency range with few parameters.
The finite element discretization of the equation of motion leads to the following matrix system:

$$\left[\mathbb{K}_e + \mathrm{i}\omega\hat{h}_G(\omega)\mathbb{K}_v^G + \mathrm{i}\omega\hat{h}_K(\omega)\mathbb{K}_v^K - \omega^2\mathbb{M}\right]\hat{\mathbf{U}} = \hat{\mathbf{F}} \tag{2}$$

where $\mathbb{K}_e = \mathbb{K}_{ep} + G_0\left(\mathbb{K}_v^G\right)_0 + K_0\left(\mathbb{K}_v^K\right)_0$, $\mathbb{K}_v^G = (G_\infty - G_0)\left(\mathbb{K}_v^G\right)_0$, and $\mathbb{K}_v^K = (K_\infty - K_0)\left(\mathbb{K}_v^K\right)_0$. The matrix $\mathbb{K}_{ep}$ is the stiffness matrix associated to the volume of elastic part of the model, $\mathbb{M}$ is the mass matrix of the whole system, and the stiffness matrix associated to the volume of viscoelastic material (computed with unitary moduli), is separated into a spheric part $\left(\mathbb{K}_v^K\right)_0$ and a deviatoric part $\left(\mathbb{K}_v^G\right)_0$. According to Equation (1), the functions $\hat{h}_G(\omega)$ and $\hat{h}_K(\omega)$ are expressed as:

$$\hat{h}_G(\omega) = \frac{\tau_G^{\alpha_G}}{(\mathrm{i}\omega)^{1-\alpha_G}[1 + (\mathrm{i}\omega\tau_G)^{\alpha_G}]} \quad \text{and} \quad \hat{h}_K(\omega) = \frac{\tau_K^{\alpha_K}}{(\mathrm{i}\omega)^{1-\alpha_K}[1 + (\mathrm{i}\omega\tau_K)^{\alpha_K}]} \tag{3}$$

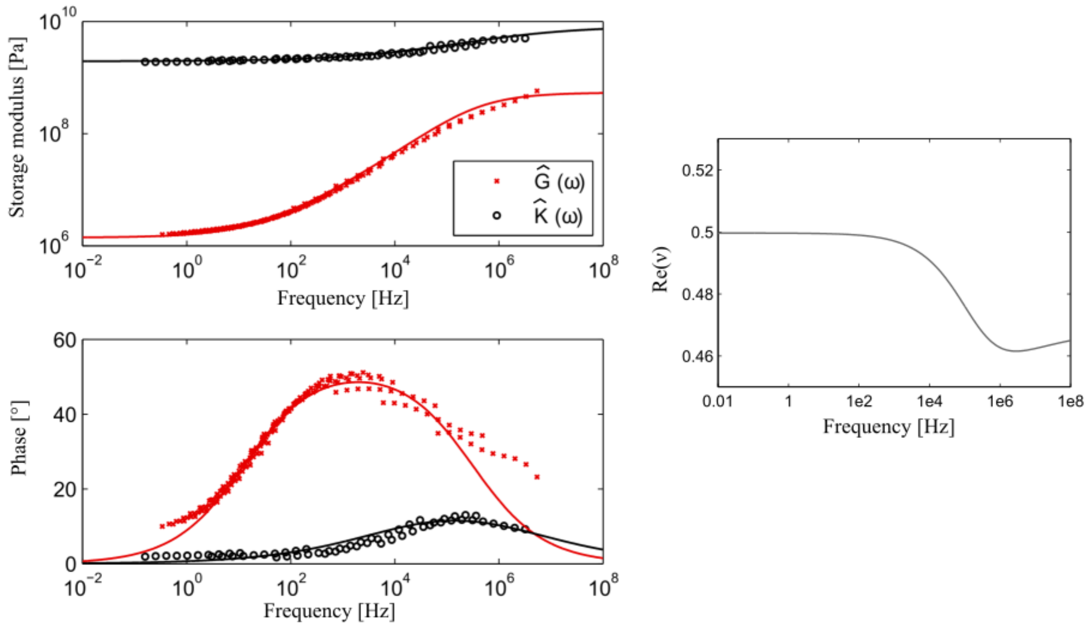

Figure 1: Master curves of the complex shear and bulk moduli of Deltane 350 (Paulstra®) measured (points) by DMA and fitted (lines) by a fractional derivative model (left). Real part of the complex Poisson ratio from identified models (right).

3 COUPLED NEWMARK-DIFFUSIVE SCHEME

Equation (2) can be rewritten in the time domain as follows:

$$\mathbb{M}\ddot{\mathbf{U}} + \left(h_G(t) \star \mathbb{K}_v^G + h_K(t) \star \mathbb{K}_v^K\right) \dot{\mathbf{U}} + \mathbb{K}_e \mathbf{U} = \mathbf{F}(t) \tag{4}$$

where the symbol $\star$ represents a convolution product.
Following e.g. [3, 5], letting $\mathbf{V} = \dot{\mathbf{U}}$, the functions $h_G(t)$ and $h_K(t)$ will be realized by a standard diffusive representation of the form:

$$\frac{\partial \boldsymbol{\varphi}(\xi, t)}{\partial t} = -\xi \boldsymbol{\varphi}(\xi, t) + \mathbf{V}(t), \quad \text{with } \boldsymbol{\varphi}(\xi, 0) = \mathbf{0} \tag{5}$$

observed through the continuous superposition:

$$(h_j \star \mathbf{V})(t) = \int_0^{\infty} \mu^j(\xi) \boldsymbol{\varphi}(\xi, t) \mathrm{d}\xi \quad \text{with } j = G, K \tag{6}$$

This exact diffusive representation can be approximated as follows:

$$\int_0^{\infty} \mu^j(\xi) \boldsymbol{\varphi}(\xi, t) \mathrm{d}\xi \approx \sum_{n=1}^{N} \mu_n^j \boldsymbol{\varphi}(\xi_n, t) \quad \text{with } j = G, K \tag{7}$$

where N is the number of approximation nodes, ξ_n a sequence of angular frequencies in the frequency range of interest and μ_n^G and μ_n^K are the corresponding optimal weights computed by minimising the respective functions $\mathcal{C}_G(\boldsymbol{\mu}^{\boldsymbol{G}})$ and $\mathcal{C}_K(\boldsymbol{\mu}^{\boldsymbol{K}})$ defined as [4]:

$$\mathcal{C}_j(\boldsymbol{\mu}^{\boldsymbol{j}}) = \sum_{l=1}^{L} \left| \sum_{n=1}^{N} \frac{\mu_n^j}{\mathrm{i}\omega_l + \xi_n} - \frac{1}{(\mathrm{i}\omega_l)^{1-\alpha_j}} \right|^2 \quad \text{with } j = G, K \tag{8}$$

where ω_l are angular frequencies and $L >> N$.
This diffusive representation is integrated into a Newmark integration scheme, by considering the functions $\varphi_n(\xi_n) := \varphi(\xi_n, t)$ as internal variables updated at each time steps. More details on the time integration scheme have been presented in [5].

4 ANALYTICAL SOLUTION FOR A SINGLE-DEGREE-OF-FREEDOM SYSTEM

The equation of motion for a single-degree-of-freedom system is:

$$\left(k_e + \frac{(\mathrm{i}\omega\tau)^\alpha}{1+(\mathrm{i}\omega\tau)^\alpha}k_v - \omega^2 m\right)\hat{u} = \hat{f}, \tag{9}$$

and can be rewritten in the time domain as:

$$\left[m\tau^\alpha(D_t)^{2+\alpha} + m(D_t)^2 + (k_e+k_v)\tau_\alpha(D_t)^\alpha + k_e\right]u(t) = \left[1+\tau^\alpha(D_t)^\alpha\right]f(t) \tag{10}$$

where $(D_t)^\beta$ represents the time derivative of order β (integer or fractional), and $\alpha = p/q$, with p and q integers satisfying $p/q \in [0,1]$.
To analytically solve this equation, the exact solution is expressed in terms of fractional power series:

$$u(t) = \sum_{n=0}^{\infty} u_n t^{\frac{n}{q}} \tag{11}$$

where the coefficients u_n are calculated from initial conditions and recurrence relationships.

5 CONCLUSION

The coupled Newmark-diffusive scheme described in this paper will be used to compute the time response of a viscoelastically damped structure, modelled by 3D finite elements. Validation of the proposed approach will be carried out on the single-degree-of-freedom system, by comparing the analytical solution with that obtained by the proposed approach.

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