EXTENSION OF THE VARIATIONAL THEORY OF COMPLEX RAYS FOR HETEROGENEOUS MEDIA

H. RIOU[1], H. LI[1] and P. LADEVEZE[1]

[1]LMT, ENS Cachan, CNRS, Université Paris-Saclay, F-94235, Cachan, FRANCE
Email: riou@lmt.ens-cachan.fr, hao-li@lmt.ens-cachan.fr, ladeveze@lmt.ens-cachan.fr

ABSTRACT

The Variational Theory of Complex Rays (VTCR) is a numerical technique that has been developed for the prediction of vibration problems in the medium frequency regime. It is a Trefftz Discontinuous Galerkin method which uses plane wave functions as shape functions. As such, one of its characteristics is the necessity for the shape functions to satisfy exactly the governing equation. For heterogeneous media, this is clearly a difficulty, as no such exact solution is known. In this paper, the VTCR is extended to bypass this difficulty, by creating a new base of shape functions.

[1] INTRODUCTION

Today, one way to efficiently solve the medium frequency problems is to adopt a Trefftz approach. By doing this, the user makes an analysis based on shape functions which satisfy exactly the governing equation, then containing a strong knowledge of the physical problem. These methods are, for example, the partition of unity method [1], the ultra weak variational method [2], the least square method [3], the plane wave discontinuous Galerkin method [4], the method of fundamental solutions [5] the discontinuous enrichment method [6], the wave based method [7]. The Variational Theory of Complex Rays (VTCR), which is the approach used in this paper, also belongs to this category of strategies. It has been introduced in [8]. All these techniques have shown a good efficiency for the resolution of vibration problems. However, they are mainly all limited to homogeneous media, i.e. to constant wave numbers.

In this paper, we propose a development of the VTCR which allows us to solve vibration problems with varying wave numbers. It is based on the definition of a new type of shape functions, composed of Airy functions, which satisfy a priori the dominant part of the governing equation.

[2] REFERENCE PROBLEM TO SOLVE

Let us consider a 2-D Helmholtz problem defined on Ω with the boundary $\partial\Omega=\partial_1\Omega\cup\partial_2\Omega$ where Dirichlet and Neumann can be prescribed. The reference problem to solve is: find $u\in H^1(\Omega)$ such that

$$\begin{aligned}&(1-i\eta)\Delta u+k^2u=0 \text{ over } \Omega\\&u=u_d \text{ over } \partial_1\Omega\\&(1-i\eta)\partial_n u=g_d \text{ over } \partial_2\Omega\end{aligned}$$

where η is the (positive) damping coefficient, k the wave number and ∂_n is the normal derivative. u_d and g_d are prescribed boundary conditions.

[3] VTCR FORMULATION OF THE REFERENCE PROBLEM

Let us suppose that Ω is partitioned in E subdomains: $\Omega=\cup_{e=1}^{E}\Omega_e$. We denote by $\Gamma_{e,e'}$ the common boundary between Ω_e and $\Omega_{e'}$, and by $\Gamma_{e,e}$ the common boundary between Ω_e and $\partial\Omega$. The VTCR strategy consists in finding the solution

$$u\in U=\left\{u/u_e\in U_e=\left\{u_e/(1-i\eta)u_E+k^2u_E=0 \text{ over } \Omega_e\right\}\right\}$$

such that

$$\begin{aligned}\mathrm{Re}\Bigg(-ik\Bigg(&\sum_e\int_{\Gamma_{e,e'}}\left(\frac{1}{2}\{q_u.n\}_{ee'}\{\bar{v}\}_{ee'}-\frac{1}{2}[\overline{q_v}.n]_{ee'}[u]_{ee'}\right)dS\\&-\sum_e\int_{\Gamma_{ee}\cap\partial_1\Omega}\overline{q_v}.n(u-u_d)\,dS+\sum_e\int_{\Gamma_{ee}\cap\partial_2\Omega}(q_u.n-g_d)\bar{v}\,dS\Bigg)\Bigg)\\&=0 \quad \forall v\in U\end{aligned}$$

where $\{u\}_{ee'}=(u_e+u_{e'})_{\Gamma_{ee'}}$, $[u]_{ee'}=(u_e-u_{e'})_{\Gamma_{ee'}}$, $q_v=(1-i\eta)\,grad\,u$. The over bar represents the complex conjugated part of a number, and Re the real part. The existence and uniqueness of solution in this kind of variational formulation have been proved in [9]. An approximated solution can be found by satisfying this variational formulation in a subspace of U of finite dimension.

[4] DEFINITION OF SHAPE FUNCTIONS

As mentioned in the introduction, we consider here the case where the wave number varies. We suppose, then, that we can write $k^2=\alpha x+\beta y+\gamma$, α, β and γ being constant parameters in Ω_e. The shape function needs to satisfy $(1-i\eta)u+k^2u=0$. It can be shown that, in such a case, the shape function are described by Airy functions. Different description can be used when selecting Airy functions. We have decided to use this description: the shape functions are described by $\Psi(x,y)=F(\tilde{x})G(\tilde{y})$, where $F(\tilde{x})=Bi(-\tilde{x})+i\,Ai(-\tilde{y})$ and $G(\tilde{y})=Bi(-\tilde{y})+i\,Ai(-\tilde{x})$, where Ai and Bi are the Airy functions. The new space variables are defined by $\tilde{x}=\dfrac{k_m^2\cos^2\theta+\alpha(x-x_m)}{\alpha^{2/3}(1-i\eta)^{1/3}}$ and $\tilde{y}=\dfrac{k_m^2\sin^2\theta+\alpha(y-y_m)}{\beta^{2/3}(1-i\eta)^{1/3}}$. k_m^2 represents the minimum value of k^2 on Ω_e and (x_m, y_m) is the coordinate which enables k^2 to take its minimum value k_m^2. θ represents the polar direction in the 2-D coordinates. Thanks to this way of doing, the selected shape functions satisfy the properties $F(\tilde{x}) \to \cos(k_1 x) + i \sin(k_1 x)$ when $\alpha \to 0$, and $G(\tilde{y}) \to \cos(k_2 y) + i \sin(k_2 xy)$ when $\beta \to 0$. Then, the shape functions tend toward propagative plane waves when the medium becomes homogeneous.

In order to have an approximated solution, one just needs to satisfy the variational formulation in a subspace U_N of U, of dimension N. The classical way to define such a subspace is to select only N direction θ_i, $\theta_i \in \{0; 2*\pi/N; \ldots; (N-1)\pi/N\}$, in the 2-D polar representation. By doing this, one naturally gets a matrix system to solve, where the matrix corresponds to the projection of the bilinear part of the variational formulation on $U_N \times U_N$, the second member its projection on U_N and the unknown vector the amplitudes of the N shape functions Ψ_i which approximate the exact solution.

[5] NUMERICAL ILLUSTRATION

We consider a simple geometry of square [0 m;1 m]×[0 m;1 m] for the domain Ω. In this domain, $\eta = 0.01$, α = 150 m^{-3}, β = 150 m^{-3} and γ = 1000 m. The selected boundary conditions are Dirichlet conditions such that the exact solution is $u_{ex} = \sum_{i=1}^{3} \Psi_i(x, y)$ with $\theta_1 = 10°$, $\theta_2 = 55°$ and $\theta_3 = 70°$. The relative error between the exact and the approximated solution is computed through $\sqrt{\int_\Omega |u - u_{ex}|^2 d\Omega / \int_\Omega |u_{ex}|^2 d\Omega}$. Three space decompositions are considered: either Ω is considered as one subdomain, or is cut in four parts, or in nine parts (see figure 1).

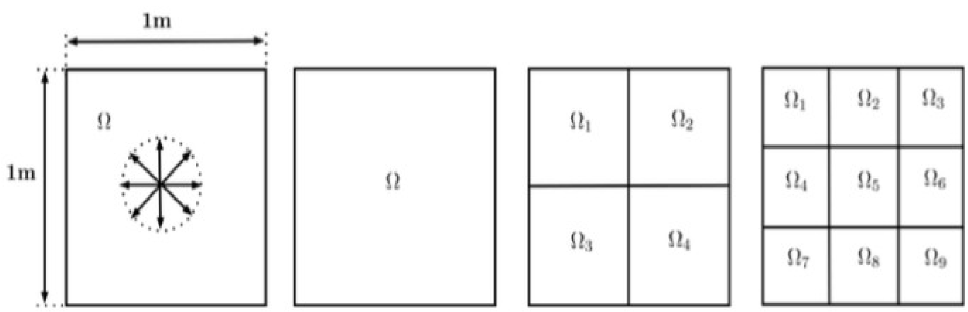

Figure 1. The considered decompositions for the selected example in Section 5.

As shown in figure 1 and explained in the last section, the approximated solutions are searched by using N shape functions regularly distributed in the 2-D polar coordinates, in each sub-domain of Ω. The convergence curve is represented in Figure 2.

As one can see, the strategy converges very fast toward the exact solution. Then, with the VTCR, few degrees of freedom are needed to get a good proximation of the solution of the reference problem. Moreover, one can see that the VTCR better works with large subdomains with many shape functions inside, than small subdomains with few shape functions inside. This behaviour has already been observed on the classic VTCR.

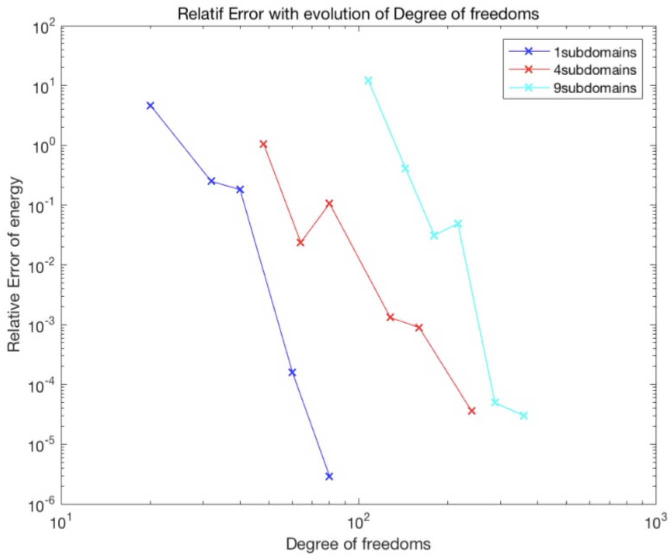

Figure 2. Convergence curves for the example considered in Section 5.

[6] CONCLUSION

In this works, we present how to use the VTCR for the resolution of heterogeneous media. The definition of a new type of shape function is done. This can be used on media where the wave number linearly varies thanks to the space. A numerical example shows that this works perfectly, and that it behaves like the classic VTCR. More complex media are the focus of our research.

REFERENCES

[1] T. Strouboulis, R. Hidajat, Partition of unity method for Helmholtz equation: q-convergence for plane-wave and wave-band local bases, Appl. Math. 51 181-204, 2006.

[2] O. Cessenat, B. Despres, Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz prob- lem, SIAM J. Numer. Anal. 35 255-299, 1998.

[3] G. Gabard, P. Gamallo, T. Huttunen, A comparison of wave-based discontinuous Galerkin, ultra-weak and least-square methods for wave problems, Int. J. Num.Methods Eng. 85 380-402, 2011.

[4] C. J. Gittelson, R. Hiptmair, I. Perugia. Plane wave discontinuous Galerkin methods: analysis of the h-version, ESAIM: Mathematical Modeling and Numerical Analysis 43 297-331, 2009.

[5] A. Barnett, T. Betcke, Stability and convergence of the method of fundamental solutions for Helmholtz problems on analytic domains, J. Comput. Phys. 227 7003-7026, 2008.

[6] C. Farhat, I. Harari, L. Franca, The discontinuous enrichment method, Comput. Methods Appl. Mech. Eng. 190 6455-6479, 2001.

[7] W. Desmet, P. Sas, D. Vandepitte, An indirect Trefftz method for the steady-state dynamic analysis of coupled vibro-acoustic systems, Comput. Assisted Mech. Eng. Sci. 8 271-288, 2001.

[8] P. Ladevèze, A new computational approach for structure vibrations in the medium frequency range, C. R. Acad. Sci. Paris 332 849-856, 1996.

[9] P. Ladevèze, H. Riou. On Trefftz and weak Trefftz discontinnuous Galerkin approaches for medium-frequency acoustics, Computer Methods in Applied Mechanics and Engineering 278 729-743, 2014.