



## **ANALYTICAL AND NUMERICAL LOCAL SENSITIVITY ANALYSIS OF PERIODIC SPRING-MASS CHAINS**

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### **ABSTRACT**

*Periodic structures can be used as mechanical filters in vibration control by creating destructive wave interferences to block the passage of propagating waves in certain frequency bands. Recently, researchers have been concentrating their effort to improve this phenomenon while creating more complex and adaptive structures. Nonetheless, there is still a lack of information about their behavior and parameters sensitivity to comprehend, for example, the effects of uncertainties. This study shows the use of analytical equations of spring-mass chains and their derivatives to compare the sensitivity of Bragg's and resonance bandgaps. The objective is to inspect the behavior and the influence of changing stiffness and inertia properties on the attenuation zones borders. The transfer matrix method is used and a general formulation is adopted to model the unit cells. The propagation constants obtained by solving eigenvalue problems and the frequency responses are used to analyze infinite and finite chains, respectively. Analytical partial derivatives and finite differences are used to calculate the local sensitivity. The bandgap borders are found to be highly sensitive to the distance from localized modes and anti-resonances. The results from the comparison between these two kinds of attenuation zones are presented.*

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## 1 INTRODUCTION

Periodic structures as chain of spring-mass systems have been studied since Newton's times [1]. More recent studies [2-3] present a complete outline about the past and recent works and the future of periodic structures. However, comparisons between the two known bandgaps types, Bragg's and resonance, are scarce. Based on these works, but with a different formulation, the present work investigates local sensitivity of bandgap borders by varying stiffness and inertia properties considering infinite and finite models. Analytical and semi-analytical solutions are used for infinite (IM) and finite (FM) models, respectively.

## 2 METHODOLOGY

The Figure 1.a) presents the general spring-mass cell. Bragg's (Figure 1.b) and resonance (Figure 1.c) bandgaps can be obtained by considering a combination of this general unit cell.

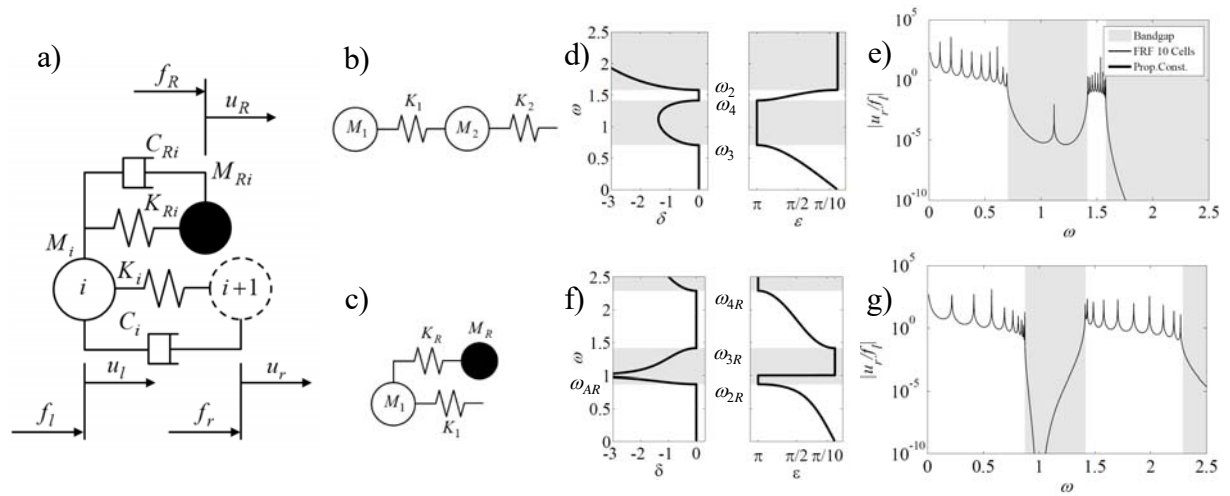


Figure 1. a) General spring-mass cell, b) Bragg's spring-mass cell , c) Resonance spring-mass cell and their corresponding propagation constants d) and f), frequency response functions e) and g).

The relation between displacements  $u$  and forces  $f$  to the left  $l$  and to the right  $r$  of the cell  $i$  is obtained by developing the equations of motion for these 2 degree-of-freedom systems. The transfer matrix method consists in characterizing the dynamics of each individual cell by its matrix  $T^{(i)}$  and combining these matrices for the computation of the behavior of assembled structures. A finite periodic structure composed of  $n$  cells is hence described by Equations (1.a) and (1.b), showing the relation between left and right degrees-of-freedom.

$$\begin{Bmatrix} u_l^{(n)} \\ f_l^{(n)} \end{Bmatrix} = \prod_{i=1}^n [T_R^{(i)}] \begin{Bmatrix} u_r^{(i)} \\ f_r^{(i)} \end{Bmatrix} \quad \text{with} \quad (1.a)$$

$$[T_R^{(i)}] = \begin{bmatrix} (1 - \Omega_i^2) - \frac{K_R}{K_i} \left( \frac{\Omega_{Ri}^2}{1 - \Omega_{Ri}^2} \right) & -\frac{1}{K_i} \\ -K_i \Omega_i^2 - K_{Ri} \left( \frac{\Omega_{Ri}^2}{1 - \Omega_{Ri}^2} \right) & -1 \end{bmatrix}. \quad (1.b)$$

The dimensionless frequencies are represented by  $\Omega = \omega/\omega_0$  and  $\Omega_R = \omega/\omega_{AR}$  with  $\omega_0 = \sqrt{K/M}$  and  $\omega_{AR} = \sqrt{K_R/M_R}$ . The springs and masses of these models are represented by  $K$  and  $M$  with the subscript  $R$  standing for resonance. For the sake of simplicity, in analytical formulation, no damping is considered; it is important to mention that the definition of bandgaps

limits is lost for high damping levels. The analytical equation of the borders can be found by calculating the eigenvalues of the transfer matrix and imposing the condition of transition between real ( $\delta$ ) and imaginary ( $\varepsilon$ ) propagation constants (Figures 1.d and 1.f). Using the same transfer matrix, but rearranging the DOF, the frequency response function (FRF) for one cell can also be found. For a finite structure, the multiplication of these matrices must be considered for each cell.

### 3 NUMERICAL EXAMPLE AND RESULTS

Table 1 shows the spring and mass values used for the numerical examples. For a Bragg's bandgap, the spring-mass cell 1 (SMC1) and 2 (SMC2) are used and for a resonance bandgap, the spring-mass cell 3 (SMC3) and 4 (SMC4) are used.

$X$	$K_1[N/m]$	$M_1[kg]$	$K_2[N/m]$	$M_2[kg]$	$K_R[N/m]$	$M_R[kg]$
SMC1 (a)	1	1	1	4	-	-
SMC2 (b)	0.5	1	1	2	-	-
SMC3 (c)	1	1	-	-	1	1
SMC4 (d)	1	2.5	-	-	1	1

Table 1. Spring and mass values for numerical examples.

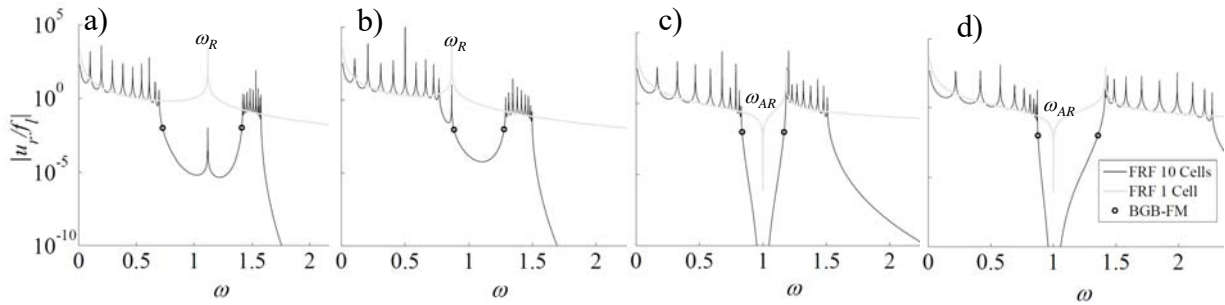


Figure 2. Frequency responses for: a) SMC1, b) SMC2, c) SMC3 and d) SMC4.

The corresponding FRFs for a finite structure with 1 and 10 cells are shown in Figure 2. After calculating the propagation constants, the following equations defining the borders and the resonance inside the bandgap, can be found:

$$\omega_2 = \sqrt{\left(\frac{\omega_4}{2}\right) - \sqrt{\left(\frac{\omega_4}{2}\right)^2 - 4 \frac{K_1 K_2}{M_1 M_2}}}, \quad \omega_3 = \sqrt{\left(\frac{\omega_4}{2}\right) + \sqrt{\left(\frac{\omega_4}{2}\right)^2 - 4 \frac{K_1 K_2}{M_1 M_2}}}, \quad (2.a)$$

$$\omega_{2R} = \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right) - \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right)^2 - 4 \frac{K_R K}{M_R M}}}, \quad \omega_{4R} = \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right) + \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right)^2 - 4 \frac{K_R K}{M_R M}}} \quad (3.a)$$

$$\omega_{2R} = \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right) - \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right)^2 - 4 \frac{K_R K}{M_R M}}}, \quad \omega_{4R} = \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right) + \sqrt{\left(2 \frac{K}{M} + \omega_{3R}\right)^2 - 4 \frac{K_R K}{M_R M}}} \quad (3.b)$$

$$\text{with } \omega_4 = \sqrt{\frac{(K_1 + K_2)(M_1 + M_2)}{M_1 M_2}} \quad \text{and} \quad \omega_{3R} = \sqrt{\frac{K_R (M_R + M)}{M_R M}}. \quad (4.a)$$

$$\omega_{3R} = \sqrt{\frac{K_R (M_R + M)}{M_R M}}. \quad (4.b)$$

$$\omega_R = \sqrt{\frac{K_1 (M_1 + M_2)}{M_1 M_2}} \quad (5.a)$$

These frequencies are represented in Figures 1.d, 1.f and 2. The derivative of these equations, in function of each parameter, gives local analytical sensitivity of bandgap borders for an infinite model (IM). For finite structures, a magnitude threshold equal to  $10^{-2}$  is used to define the bandgap borders (BGB) using frequency response functions (FRF) as presented in Figure 2. In this case, finite differences are used to calculate the local sensitivity. The results are in Figure 3.

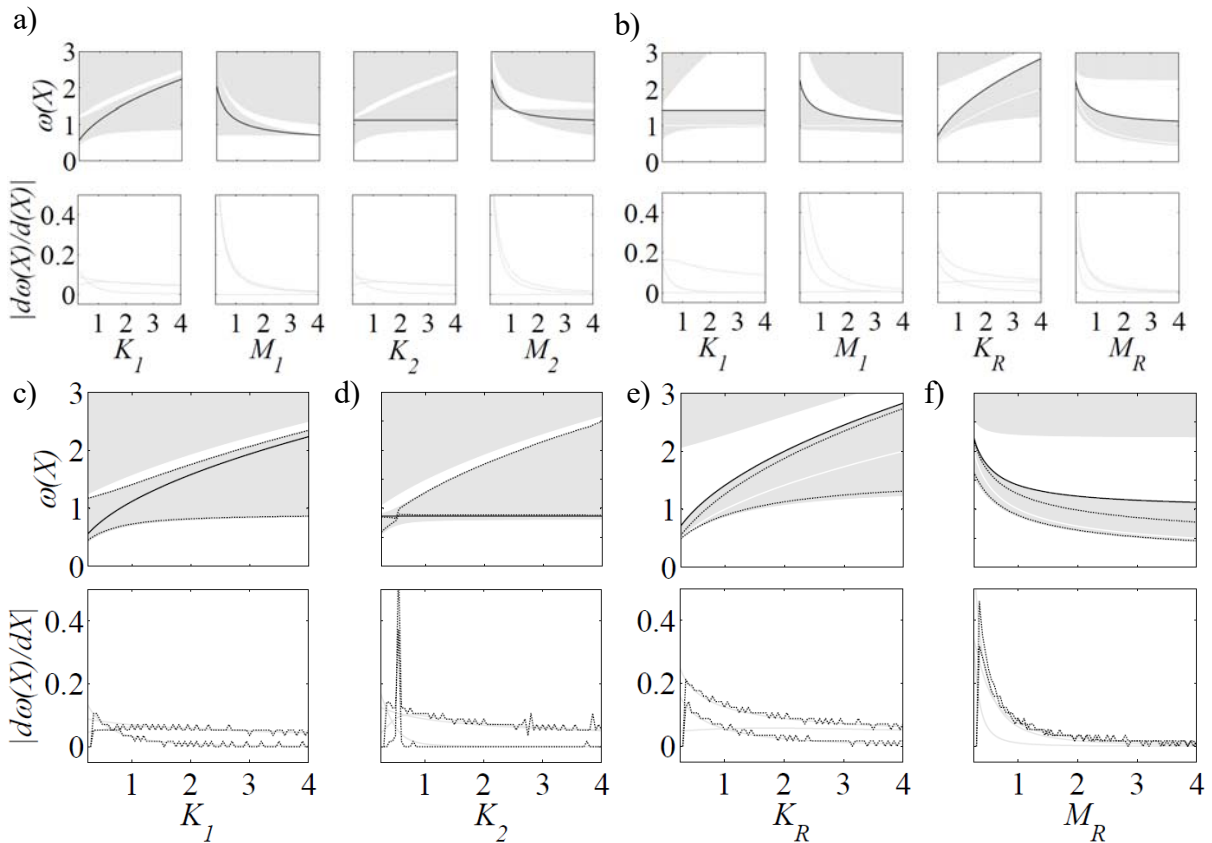


Figure 3. Varying spring and mass values and their analytical derivatives for a) SMC1 and b) SMC2, varying specified parameters for c) SMC1, d) SMC2, e) SMC3 and f) SMC4 with bandgap envelope for infinite model (■), its borders analytical derivatives (—), finite model borders and their finite differences (.....), resonances (—) and anti-resonances (—).

Figure 3.d shows that a resonance inside a bandgap can reduce the bandgap width depending on the number of cells for finite structures. Similarly, in Figures 3.e and 3.f, the anti-resonance does the same effect if it is located far from a bandgap border.

#### 4 CONCLUSION

A simple transfer matrix model of a general spring-mass cell was used. The analytical solution for bandgap borders of Bragg's and resonance attenuation zones were presented. The local sensitive was found analytically by calculating the derivative and numerically by using finite differences. The behaviour for bandgap borders of Bragg's and resonance attenuation zones were compared. The resonances inside the bandgaps, also known as localized modes, and the anti-resonances can change the value for a bandgap border in a finite model.

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