STUDY OF THE FIRST BRAGG BAND GAP FEATURES OF EULER CONTRASTED PERIODIC BEAMS

Adrien Pelat[1] , Thomas Gallot [2] and François Gautier[1]

[1]Laboratoire d'Acoustique de l'Universit du Maine
Universit du Maine, Avenue Olivier Messiaen, 72085 cedex 9 , Le Mans, FRANCE or address
Email: adrien.pelat@univ-lemans.fr , francois.gautier@univ-lemans.fr

[2]Instituto de Fisica de la Facultad de Ciencias,
Universidad de la Republica, Igua 4225, Montevideo, URUGUAY 11400
Email: tgallot@fisica.edu.uy

ABSTRACT

The vibration control of lightweight structures is nowadays a challenge of great industrial interest because of obvious ecological and economical reasons. Among the possible strategies, applying the concepts of metamaterials to the vibro-acoustics context seems to be promising. It can be done by designing the structures as a periodic distribution of a unit cell. The overall properties of such structures then result from a carefull design of the mechanical properties and possible resonances of the unit cell. This work deals with beams made of uniform material and with continuously graded flexural rigidity driven by variable thickness. The study focuses on the first Bragg band gap of such structures by means of both theoretical and experimental approaches. Particularly, explicit relations linking the properties contrast with the band gap width and central frequency are derived in an ideal case of a hollow beam without flanks for which a PWE model can be analytically solved. The theoretical results obtained in this ideal case succesfully match both the numerical results obtained from a PWE method and experimental measurements.

1 INTRODUCTION

From works of Brillouin on wave propagation in periodic media [1], meta-materials has been widely studied during the last decades in many fields of physics. The typical effects obtained with meta-materials can be exploited in three main categories of applications : wave filtering, wave collimation and cloaking.

When applied to the context of vibro-acoustics, the case of flexural waves is of particular interest when dealing with structure borne sound from shells or plates. Wave filtering effects can then be very usefull to get non resonant and so non radiating structures without added mass in given frequency bandwidths [2]. Then, providing general design rules of such meta-structures is of great interest and still an open question [3]. For example, no explicit link between band gap features and structure geometrical or material properties is well known.

As a preliminary work, the study of academic beams is often useful in order to apprehend the practical and more complicated case of plates. In this work, the aim is to study how the thickness contrast of a continuously varying periodic beam is driving the Bragg band gap central frequency and bandwidth.

After defining a general PWE formalism of Euler beams in section 2, the ideal case of a hollow rectangular beam is presented in section 3. Theoretical, numerical and experimental results are then compared and discussed in section 4.

2 PWE GENERAL FORMALISM FOR AN EULER-BERNOULLI BEAM

Under Euler-Bernoulli assumtions and considering harmonic motion ($e^{j\omega t}$), the free flexural displacement $w(x)$ in a beam of variable height $h(x)$ and constant width b satisfies to the equation of motion

$$-\rho h(x)\omega^2 w(x) + \frac{\partial^2}{\partial x^2}\left(D(x)\frac{\partial^2 w(x)}{\partial x^2}\right) = 0, \tag{1}$$

where $\rho h(x)$ is the surface mass with ρ the material volumic mass, $D(x) = \dfrac{Eh(x)^3}{12}$ is the surface flexural rigidity with $E = E_0(1 + j\eta)$ the material complex young modulus in which E_0 is the elastic constant and η is the loss factor.

According to the plane wave expansion method, the solutions of equation (1) are sought as the following series

$$w(x) = \sum_{g_1} w_{g_1}(k)e^{jg_1 x}e^{jkx}, \tag{2}$$

with k a given flexural wavenumber, $g_1 = \dfrac{n_1 2\pi}{L}$ with n_1 an integer.

Considering the beam being a periodic distribution of a unit cell of size L, the mechanical properties can be expanded as the following Fourier series :

$$\rho h(x) = \sum_{g_2} \alpha_{g_2} e^{jg_2 x} \quad \text{and} \quad D(x) = \sum_{g_2} \delta_{g_2} e^{jg_2 x} \tag{3}$$

where $\alpha_\kappa = \dfrac{1}{L}\int_0^L \rho h(x)e^{-j\kappa x}dx$, $\delta_\kappa = \dfrac{1}{L}\int_0^L D(x)e^{-j\kappa x}dx$, and $g_2 = \dfrac{n_2 2\pi}{L}$ with n_2 an integer.

Truncating the Fourier series (3) with $n_2 \in [-N_2; N_2]$ and the plane wave expansion (2) with $n_1 \in [-N_1; N_1]$, the equation of motion (1) turns to a matrix equation

$$(P(k) - \omega^2 Q)\mathbf{W} = \mathbf{0}, \tag{4}$$

where $\mathbf{W}^t$ is a $[2N_1 + 1 \times 1]$ column vector, Q and P are $[2N_3 + 1 \times 2N_1 + 1]$ matrices with $N_3 = N_1 + N_2$ given by $Q_{n_3 n_1} = \alpha_{g_3 - g_1}$ and $P(k)_{n_3 n_1} = \delta_{g_3 - g_1}(k + g_1)^2(k + g_3)^2$.

3 ANALYTICAL DERIVATION OF THE FIRST BAND GAP WIDTH OF A HOLLOW RECTANGULAR BEAM

An ideal geometry sketched in figure 1 is now defined in order to simplify the matrices in equation (4) and then analytically solve the problem. The goal is to obtain an algebraic expression of the first Bragg band gap, defined as the difference between the two first eigenvalues at $k = \pi/L$.

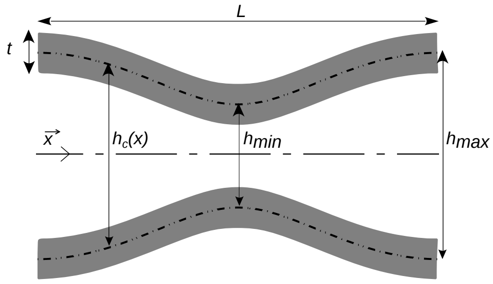


Figure 1: Profile view of the unit cell geomtry of the modeled rectangular hollow beam with no flanks. The constant width b is in the out-of-plane direction.

First, the beam cross-section is considered as a hollow rectangle where the lateral walls have been removed. The thickness t and width b of both top and bottom walls are constant while the cross-section height varies, as represented in figure 1.

Consequently, the surface mass $\rho h(x) = \rho 2t$ remains constant and so the property gradient is only carried by the varying surface flexural rigity $D(x) = \frac{Et}{2} h_c^2(x)$ whith $h_c(x)$ the central of both top and bottom wall.

Second, the spatial shape of beam unit cell profile is chosen as follows to make the flexural rigidity proportional to a cosine fonction :

$$h_c(x) = h_0 \sqrt{1 + C.cos\left(\frac{2\pi x}{L}\right)} \quad => \quad D(x) = \frac{Et}{2} h_0^2 \left[1 + C.cos\left(\frac{2\pi x}{L}\right)\right] , \tag{5}$$

where $h_0 = \sqrt{\frac{h_{max}^2 + h_{min}^2}{2}}$ is the equivalent central height for a uniform beam (same cross-section with constant height), and C is the so called contrast parameter defined as

$$C = \frac{h_{max}^2 - h_{min}^2}{h_{max}^2 + h_{min}^2}. \tag{6}$$

with h_{max} and h_{min} the maximum and minimum central height, respectively.

From this ideal geometry, the Fourier series in equations (3) for the surface mass and flexural rigidity leave a single non zero term ($N_2 = 0$) and only three non zero terms ($N_2 = 1$), respectively. Finally, it can be shown that the matrix problem (4) can be rewritten as a classical eigenvalue problem with a tridiagonal matrix $M = P(k) - \omega^2 Q$ for which the determinant can be found from a recurence equation.

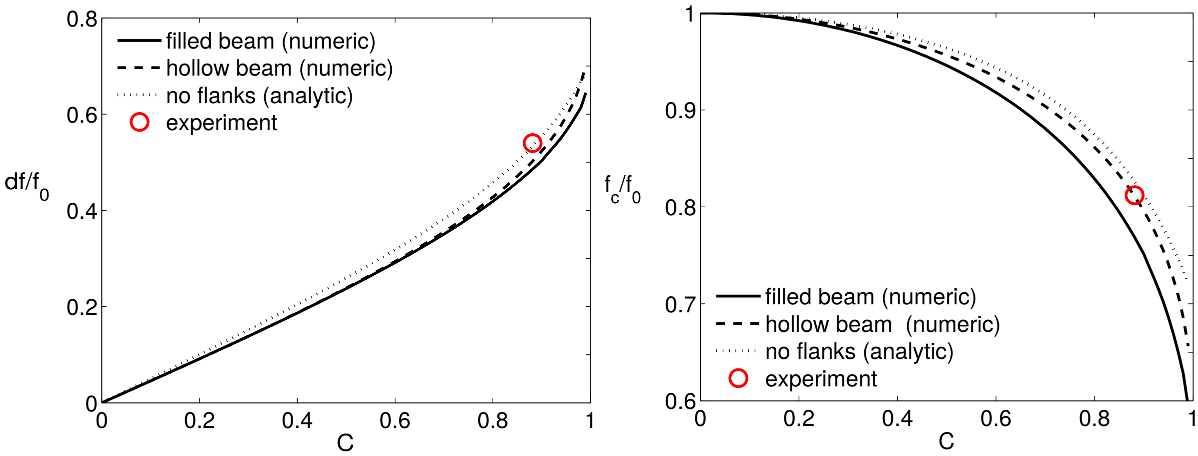
Assuming that the field plane wave expansion is truncated with $N_1 = 2$ (compromise between accuracy and convenience of analytical calculations) and after few algebra leading to cancel the matrix determinant, the gap relative bandwidth and central frequency are found to be only function of the thickness contrast C :

$$\frac{\mathrm{d}f}{f_0} \approx \frac{C}{2} \left(\frac{1 - C^2/2}{1 - 3C^2/4}\right)^{1/2} \quad ; \quad \frac{f_c}{f_0} \approx \left(\frac{1 - C^2/2}{1 - 3C^2/4}\right)^{-1/2} , \tag{7}$$

whith $f_0 = \frac{\pi}{2L^2} \sqrt{\frac{E h_0^2}{4\rho}}$.

4 RESULTS AND DISCUSSION

A thickness contrast variation is presented in figure 2. Figure 2(a) and 2(b) display the gap relative bandwidth and central frequency, respectively. Equations (7) corresponding to the ideal case of hollow beam with no flanks are plotted in dotted lines. The cases of the rectangular hollow beam (dashed lines) and fully filled rectangular beam (full line) are obtained from numerical resolution of equation (4).



(a) (b)

Figure 2: (a) First Bragg band gap bandwidth and (b) central frequency versus contrast parameter C shows no fundamental difference between cross-section geometries. Analytical results are in agreement with full PWE numerical solutions and an experimental results of an aluminium beam.

All results give same overall behavior : when the contrast increases, the gap is enlarged and shifted to low frequencies. The sligth discrepancies between the results make the no flanks assumption valid for giving a predictive analytical formula of the gap bandwidth of a fully filled contrasted real beam.

An experimental validation has also been performed with an aluminium beam. The beam geometry has been generated with a classical cutting machine. The relative gap width and position are in agreement with both analytical and numerical computation (red circle in Fig. 2).

This work demonstrates, in the case of flexural unidimensional wave, how the first frequency gap is fully characterized by a unique contrast parameter. Numerical simulation suggests that the influence of the cross-section geometry is weak. Finally, analytical expressions of gap bandwitdh and central frequency are derived and can be taken as benchmark for bandgap design.

REFERENCES

[1] Leon Brillouin. *Wave propagation in periodic structures: electric filters and crystal lattices*. Courier Corporation, 2003.

[2] DM Mead. Wave propagation in continuous periodic structures: research contributions from southampton, 1964–1995. *Journal of sound and vibration*, 190(3):495–524, 1996.

[3] Niels Olhoff, Bin Niu, and Gengdong Cheng. Optimum design of band-gap beam structures. *International Journal of Solids and Structures*, 49(22):3158 – 3169, 2012.